

# Help Notes

25 août 2017

## **Why am I asking you to justify all your answers and be precise ? Is it to annoy you ?**

Well, I am asking you to justify all your answers and be precise. Because, first, I am convince you all smart enough to succeed in doing so.

No, it is not to annoy you. It should hopefully help you. Indeed, in life, it is rare that you can just state something and that all people around you will just believe you without asking questions.

If you have a strong argument with you and you know how to construct it so good that nobody could really contradict you, then you can maybe move forward.

Math is to me the subject, where you learn how to exercise your mind/logic with strange symbols and terminologies and make you able to justify your self in a discipline way. This might help you to structure better your thoughts in any other subjects : from physics, to engineering, Philosophy...

Who cares that the number you found is 2 or 4 or 100000 if you do not tell us where it came from. I could have asked a computer it will give it to me. I care to make sure that you understand and you can tell me about how you found the number because this a computer might not be always able to do it, and maybe you had some so clever idea that I want to know about. I do not want you to believe all I say, I want you to understand what I say and I want you to be understood, because that should matter ? You are not a calculator but a human being after all.

Do not worry I will not ask any hard proof question on the tests only questions we have seen in class or during the tutorial. If you have any difficulty with the material I am here to help.

## Justify your self : prove or disprove

- When you are sure that a **statement is correct** you will want to **prepare to do a proof**. What does it mean, you want to explain me why you think the statement is correct. If your proof is good, usually what happens is that all the steps of the proof are connected to each other in a correct way. Meaning that if you write two lines : everyone should understand how you pass from one to the other and also how they are related.

Let's me take an example :

*If you want to prove : "If I take the square of an odd number then the remainder of this square in the division by 4 is 1." and I answer,  $(2k+1)^2 = 4k^2 + 4k + 1$ . Then, I believe not many of us could get my idea, right? But if I go slowly and I write : "Let's take an odd number  $n$ , I know I can always write it as  $n = 2k + 1$  where  $k$  is an integer (think about it if you need to). Now, we take the square of  $n$*

$$n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 4(k^2 + k) + 1$$

*We see that  $k^2 + k$  is an integer, so by definition of the remainder we obtain that 1 is the remainder of the division by 4. Proving the statement.* Where you are able to follow the second argument better than the first? Well if so it is maybe that you understand me better if I give you all the steps, so if you want me to understand you better give me all the steps you used to get to the answer. Do not hesitate to write sentences in English, do not write too much unnecessary stuff it could make the proof confusing just write what anyone of us needs to understand you no matter how easy/hard is this class to us. Do not forget, I do not read minds.

Unfortunately, writing good proof is hard. The learning process is very long and it will take time. So do not worry if you struggle with this, it is normal we all struggle. Do your best and at least try. Make sure your answer is correct. Ask me/tutors/friends for help. One day you will get it and you will understand that it is much better when you know how to explain yourself just right.

For proving there is not really a sample exercise, or a rule that just apply for everything. The only way to get better at it is to practice, doing exercises.

- When you are sure that a **statement is wrong**, you will need to **find a SPECIFIC counter example**. YOU DO NOT NEED MORE THAN ONE COUNTER EXAMPLE TO DISPROVE A STATEMENT.

Let's me take an example :

*If you want to disprove : " $2(x+2)^2 + 3 = 2x^2 + 5$ ", DO NOT WRITE : " $2(x+2)^2 + 3 = 2x^2 + 8x + 11 \neq 2x^2 + 5$ " because that is not correct, indeed, for  $x = 1/4$ , then  $2x^2 + 8x + 3 = 2x^2 + 5$  so what I wrote is wrong. Since I only need ONE counter example, I rather should choose ONE specific  $x$  that disprove my statement, I do not need to find more than one? So let us take  $x = 0$ ,  $2(x+2)^2 + 3 = 11$  and  $2x^2 + 5 = 5$ , thus for  $x = 0$ , we have  $2(x+2)^2 + 3 \neq 2x^2 + 5$  which disprove the statement*

One other thing, choose a counter example that makes your life easy and avoid computational mistake. What do I mean, take a counter example that give you the easiest computation possible (if possible), it could avoid you to make a mistake and thus to lose credit for your test. Be smart.

## Fixing notations

Symbol	Meaning
$\forall$	for all, for every ;
$\exists$	there exists (at least one) ;
$\exists!$	there exists exactly one ;
<i>s.t.</i>	such that ;
$\Rightarrow$	implies ;
$\Leftrightarrow$	if and only if ;
$x \in A$	the point $x$ belongs to a set $A$
$x \notin A$	the point $x$ does not belongs to the set $A$
$\mathbb{N}$	the set of natural number (counting numbers) $1, 2, 3, \dots$
$\mathbb{Z}$	the set of all the integers (positive, negative or zero)
$\mathbb{Q}$	the set of rational numbers
$\mathbb{R}$	the set of real numbers
$\mathbb{C}$	the set of complex numbers
$\{x \in A : P(x)\}$	the subset of the elements $x$ in a set $A$ such that the statement $P(x)$ is true
$\emptyset$	the empty set, the set with nothing in it
$A \subseteq B$	$A$ is a subset of $B$ i.e. any element of $A$ also belongs to $B$ (in symbolic notation : $x \in A \Rightarrow x \in B$ that we use when doing proofs).
$A = B$	the sets $A$ and $B$ contain exactly the same points This statement is equivalent to saying : $A \subseteq B$ AND $B \subseteq A$
$\{p\}$	singleton set : the set that contains only the point $p$ (Logically speaking, the "point $p$ " is not the same as the set $\{p\}$ whose only element is $p$ .)
$A \cap B$	indicates the intersection of two sets ; An element $x$ is in $A \cap B \Leftrightarrow x \in A$ AND $x \in B$ . Notice $A \cap B = B \cap A$ .
$A \cup B$	the union of two sets. An element $x$ lies in $A \cup B \Leftrightarrow$ Either $x \in A$ OR $x \in B$ (or BOTH). Notice $A \cup B = B \cup A$ .
$A \setminus B$	the set of the elements of $A$ that are not in $B$

## Introducing functions

**Domain :** In order for you to understand a function, I need to tell you where the  $x$  value of my function belongs so that you know where to pick them, that is why I give you the domain of your function, that is exactly all the value of  $x$  you are allowed to take for this function.

**Codomain** I also need to tell you where we end up when we apply this function, I do not want to have an headache and think too much about it, so usually, we will choose anything big enough so that we sure it contains all those values that I could get when I apply the function, that is the codomain.

**Rule** Finally, I obviously need to give you the rule that will create my function. This rule by definition of a function can only assign one  $f(x)$  to a given  $x$  (no more than 1).

When I write

$$\begin{array}{ccc} f: & A & \rightarrow B \\ & x & \mapsto f(x) \end{array}$$

or sometimes

$$f: A \rightarrow B \text{ defined by } f(x) = \dots$$

it is just a short way to give you all I said before : that is what I mean :

- **A is the domain :** the value of  $x$  I have chosen for you such that if I do  $f(x)$  in a calculator it will be able to return me the  $f(x)$  without an error message.

Well, maybe I do not want to give this to you in which case I could ask you to figure it out.

**Here two sample questions I could ask you about the domain :**

1. *Let*

$$\begin{array}{ccc} f: & \mathbb{R}^+ & \rightarrow \mathbb{R} \\ & x & \mapsto \sqrt{x} \end{array}$$

*What is the domain of  $f$  ?*

**If you understood me, you will answer :** *The domain of  $f$  is  $\mathbb{R}^+$ .*

2. *We want to think of a real function whose rule is given by  $f(x) = \frac{x+2}{x+3}$ , could you give me what is the maximal subset of  $\mathbb{R}$  (the real number), I can choose to define this function.*

**If you understood me, you will answer :** *This function is a quotient of two functions which are polynomials, so the only problem I can see is that the denominator should not be zero, that is the only possible problem for the function not to be defined. But  $x + 3 = 0 \Rightarrow x = -3$ , thus the maximal domain for  $f$  in  $\mathbb{R}$  is  $\mathbb{R} \setminus \{-3\}$  (that is all the real numbers but  $-3$ .)*

- **B is the codomain :** a set that contains all the values  $f(x)$  when you pick  $x$  on the domain. It is most of the time bigger than the set of all those values. (I have just to make sure when I choose the codomain that all the possible values of  $x$  belong to this set.)

**Here a sample question I could ask you in relation about the codomain :**

*Let*

$$\begin{array}{ccc} f: & \mathbb{R}^+ & \rightarrow \mathbb{R} \\ & x & \mapsto \sqrt{x} \end{array}$$

*What is the codomain of  $f$  ?*

**If you understood me, you will answer :** *The codomain of  $f$  is  $\mathbb{R}$ .*

- **$f(x)$  is the rule** : that is the recipe you need in order to be able to give me the output if I pick a point in the domain and give it to you.

If you think hard about it you will understand that if you really want to understand fully your function, you need to know all this things : domain, codomain, rule. If I change the domain or the codomain or the rule, I am indeed changing the function and thus its properties.

From this, you can now say when **two functions are the same**, they should be the same if they have same domain same codomain and same rule, right ?

In symbol terms, you get if  $f : A \rightarrow B$  and  $g : C \rightarrow D$  are two functions,  $f$  and  $g$  are the same/equal (sometimes we write  $f \equiv g$ ) if  $A = C$ ,  $D = B$  and  $f(x) = g(x)$ ,  $\forall x \in A = C$ .

**Here sample questions I could ask you about functions being equal :**

1. Do you think the function  $f : \mathbb{R} \rightarrow \mathbb{R}$ ,  $x \mapsto x$  is equal to the function  $g : \mathbb{R} \rightarrow \mathbb{R}$ ,  $x \mapsto \sqrt{x^2}$  ?  
**If you understood me, you will answer :** Well, when I try  $x = -1$ , I get  $f(-1) = -1$  and  $g(-1) = 1$ , so no those functions are not equal.
2. Do you think the function  $f : \mathbb{R}^+ \rightarrow \mathbb{R}$ ,  $x \mapsto x$  is equal to  $g : \mathbb{R}^+ \rightarrow \mathbb{R}$ ,  $x \mapsto \sqrt{x^2}$  ?  
**If you understood me, you will answer :** I can see both of those functions have same domain  $\mathbb{R}^+$  and same codomain  $\mathbb{R}$ . Also  $g(x) = \sqrt{x^2} = x = f(x)$ . So yes, those two functions are the same.

In order, to be able to speak together about those functions, let's agree with some terminologies it will help us to understand each other :

1. We will name  $f(x)$  **the image of  $x \in A$  by  $f$** . Note that  $x$  is an element of the domain and  $f(x)$  belongs to the codomain.
2. If I give you a  $y$  in the codomain of  $f$  such that there is a  $x$  in the domain of  $f$  with  $y = f(x)$ , we will say that  $x$  **is a preimage of  $y$  by  $f$** . Note I say "a" and not "the" preimage, indeed remember that a function could have more that one preimage and if I said "the" preimage I could be wrong be it would mean that I know that there is only one !

**Here sample questions I could ask you about images and preimages :**

Let  $g : \mathbb{R} \rightarrow \mathbb{R}$  defined by  $g(x) = x^2 + 3$

1. Find the image of 0,  $\sqrt{2} + 1$  by  $g$ .  
**If you understood me, you will answer :** The image of 0 is  $g(0) = 0^2 + 3 = 3$ .  $g(\sqrt{2} + 1) = (\sqrt{2} + 1)^2 + 3 = \sqrt{2}^2 + 2\sqrt{2} + 1 + 3 = 2\sqrt{2} + 6$ . Thus the image of  $\sqrt{2} + 1$  is  $g(\sqrt{2} + 1) = 2\sqrt{2} + 6$ .  
**DO NOT FORGET TO ANSWER THE QUESTION, MAKE SURE I KNOW WHAT IN YOUR COMPUTATION IS INDEED THE IMAGE.**
2. Find all the preimages of 0, 3 by  $g$ .

**If you understood me, you will answer :** The preimages of 0 are the  $x \in \mathbb{R}$  such that  $g(x) = 0$ . But,

$$g(x) = 0 \Leftrightarrow x^2 + 3 = 0 \Leftrightarrow x^2 = -3$$

But a square is never negative, thus such a  $x$  cannot exist, that means that 0 has no preimage. The preimages of 3 are the  $x \in \mathbb{R}$  such that  $g(x) = 3$ . But,

$$g(x) = 3 \Leftrightarrow x^2 + 3 = 3 \Leftrightarrow x^2 = 0$$

That is equivalent to  $x = 0$ . Thus, 3 has only one preimage which is 0.

## Operation on functions

If you have two functions with same domain and codomain you can sum them, do a difference, a product or sometimes a quotient.

Here how, let

$$\begin{array}{ccc} f: & A & \rightarrow B \\ & x & \mapsto f(x) \end{array}$$

and

$$\begin{array}{ccc} fg: & A & \rightarrow B \\ & x & \mapsto g(x) \end{array}$$

You can define :

1. the sum

$$\begin{array}{ccc} f + g: & A & \rightarrow B \\ & x & \mapsto (f + g)(x) = f(x) + g(x) \end{array}$$

2. the difference

$$\begin{array}{ccc} f - g: & A & \rightarrow B \\ & x & \mapsto (f - g)(x) = f(x) - g(x) \end{array}$$

3. the product

$$\begin{array}{ccc} fg: & A & \rightarrow B \\ & x & \mapsto (fg)(x) = f(x)g(x) \end{array}$$

4. sometimes the quotient, only if you make sure that the denominator you choose is never 0 for any value of the domain. So, first check that  $g(x) \neq 0, \forall x \in A$  and then you can define :

$$\begin{array}{ccc} \frac{f}{g}: & A & \rightarrow B \\ & x & \mapsto \frac{f}{g}(x) = \frac{f(x)}{g(x)} \end{array}$$

You can of course do more than one operation at the time and also between more than two functions.

**Here sample questions I could ask you about operations :**

Let

$$\begin{array}{ccc} f: & \mathbb{R} & \rightarrow \mathbb{R} \\ & x & \mapsto 2x + 3 \end{array}$$

and

$$\begin{array}{ccc} g: & \mathbb{R} & \rightarrow \mathbb{R} \\ & x & \mapsto 3x - 5 \end{array}$$

1. Describe the function  $f + g$ .

**If you understood me, you will answer :** The function  $f + g$  has domain  $\mathbb{R}$ ,  $\mathbb{R}$  and rule  $f + g(x) = f(x) + g(x) = 2x + 3 + 3x - 5 = 5x - 2$ .

2. Describe the function  $f - g$ .

**If you understood me, you will answer :** The function  $f - g$  has domain  $\mathbb{R}$ ,  $\mathbb{R}$  and rule  $f - g(x) = f(x) - g(x) = 2x + 3 - (3x - 5) = -x + 8$ .

3. Describe the function  $fg$ .

**If you understood me, you will answer :** The function  $fg$  has domain  $\mathbb{R}$ ,  $\mathbb{R}$  and rule  $fg(x) = f(x)g(x) = (2x + 3)(3x - 5) = 6x^2 - 10x + 9x - 15 = 6x^2 - x - 15$ .

4. Can you define the function  $\frac{f}{g}$  if so describe it.

**If you understood me, you will answer :** No we cannot because  $g(x)$  can be 0 when  $x = 5/3$  and dividing by 0 is never allowed.

## Composing functions

A bit more difficult the composition, you cannot always compose two functions you need some compatibility condition. If you take two functions

$$f: A \rightarrow B$$

and

$$g: C \rightarrow D,$$

you will only be able if **compose  $g$  with  $f$**  if and only if  $B = C$ , in other words the codomain of  $f$  is equal to the domain of  $g$ , when this is the case then you get a function  $g \circ f: A \rightarrow D$  defined by the rule  $g \circ f(x) = g(f(x))$ .

If it can help you to figure out faster the domain and the codomain of  $g \circ f$  use the following trick, if does not help just use whatever trick is best for you :

As you see we have  $g \circ f(x) = g(f(x))$ , so when you pick  $x$  this  $x$  has to belong to the domain of  $f$  since you will have to be able to compute  $f(x)$ , but then since you doing  $g(f(x))$ , in order for this to make sense you need that  $f(x)$  belongs to the domain of  $g$  otherwise how can you compute  $g(f(x))$ ? Let me draw a maybe helpful diagram for you.

$$\begin{array}{ccccccc} f \circ g: & A & \rightarrow & B = C & \rightarrow & D \\ & x & \mapsto & f(x) & \mapsto & g(f(x)) \end{array}$$

This is just symbolizing what I just say. When you do this composition rule you pick  $x \in A$ , you apply the rule of  $f$ ,  $f(x)$  then the one of  $g$  on  $f(x)$  and get  $g(f(x)) = g \circ f(x)$ . That also might help you understand why do you need  $B = C$ .

Note that you might composing  $f$  with  $g$  is a completely different function, first this one only exists if  $D = A$  and if this is the case you can define this function  $f \circ g: C \rightarrow B$  with the rule  $f \circ g(x) = f(g(x))$ .

Note that  $f \circ g$  is in general not equal to  $g \circ f$ . Even worse sometimes  $f \circ g$  exist and  $g \circ f$  does not or vice versa.

**Here sample questions I could ask you about operations :**

1. Let

$$\begin{array}{ccc} f: & \mathbb{R} & \rightarrow \mathbb{R} \\ & x & \mapsto x^2 \end{array}$$

and

$$\begin{array}{ccc} g: & \mathbb{R} & \rightarrow \mathbb{R} \\ & x & \mapsto 3x - 5 \end{array}$$

(a) Describe  $g \circ f$  if possible.

**Solution :** Since the codomain  $\mathbb{R}$  of  $f$  is equal to the domain of  $g$ ,  $g \circ f$  exist and is define as  $g \circ f: \mathbb{R} \rightarrow \mathbb{R}$  with the rule  $g \circ f(x) = g(f(x)) = g(x^2) = 3x^2 - 5$ .

(b) Describe  $f \circ g$  if possible.

**Solution :** Since the codomain  $\mathbb{R}$  of  $g$  is equal to the domain of  $f$ ,  $f \circ g$  exist and is define as  $f \circ g: \mathbb{R} \rightarrow \mathbb{R}$  with the rule  $f \circ g(x) = f(g(x)) = g(3x - 5) = (3x - 5)^2$ .

(c) Is  $g \circ f \equiv f \circ g$  ?

**Solution :** Note that for  $x = 0$ ,  $f \circ g(x) = (3 \times 0 - 5)^2 = 25$  and  $g \circ f(x) = 3 \times 0^2 - 5 = -5$ . Thus  $f \circ g(0) \neq g \circ f(0)$  (Remember what we said earlier to disprove an argument we give a counter example).

2. Let

$$\begin{array}{ccc} f: & \mathbb{R} & \rightarrow \mathbb{R} \\ & x & \mapsto x^2 \end{array}$$

and

$$\begin{array}{ccc} g: & \mathbb{R}^+ & \rightarrow \mathbb{R} \\ & x & \mapsto \sqrt{x} \end{array}$$

(a) Describe  $g \circ f$  if possible.

**Solution :** Since the codomain  $\mathbb{R}$  of  $f$  is not equal to the domain of  $g$   $\mathbb{R}^+$  we cannot compose  $g$  with  $f$ , thus  $g \circ f$  does not exist.

(b) Describe  $f \circ g$  if possible.

**Solution :** Since the codomain  $\mathbb{R}$  of  $g$  is equal to the domain of  $f$ ,  $f \circ g$  exist and is define as  $f \circ g: \mathbb{R}^+ \rightarrow \mathbb{R}$  with the rule  $f \circ g(x) = f(g(x)) = f(\sqrt{x}) = (\sqrt{x})^2 = x$ .

(c) Is  $g \circ f \equiv f \circ g$ ?

**Solution :** No, one exists and the other one even does not exist.

3. If possible, write

$$\begin{array}{ccc} t: & \mathbb{R} & \rightarrow \mathbb{R} \\ & x & \mapsto 3x^2 + 5 \end{array}$$

as the a composite  $f \circ g$  where

$$\begin{array}{ccc} g: & \mathbb{R} & \rightarrow \mathbb{R} \\ & x & \mapsto x^2 \end{array}$$

**Solution :** Since  $t$  has domain  $\mathbb{R}$  and codomain  $\mathbb{R}$ , we need that the domain of  $g$  is equal to the domain of  $t$   $\mathbb{R}$  and the codomain of  $f$  is equal to the codomain of  $t$  thus also  $\mathbb{R}$ . Moreover since the codomain of  $f$  has to be equal to for the composition to be defined, you need to have the codomain of  $f$  equal to  $\mathbb{R}$  and  $f: \mathbb{R} \rightarrow \mathbb{R}$  is defined by the rule  $f(x) = 3x + 5$ . We can check that  $f \circ g: \mathbb{R} \rightarrow \mathbb{R}$  is defined by  $f \circ g(x) = f(x^2) = 3x^2 + 5 = t(x)$ .

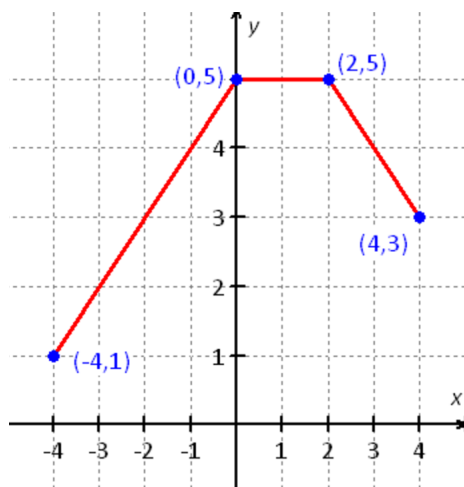
## Graph of a function

The **Graph of a function**  $f : A \rightarrow B$  is a set of points with coordinates  $(x, f(x))$ . You write this in mathematical symbols as

$$\text{Graph}(f) = \{(x, f(x)) \in A \times B : x \in A\}$$

When you have a function with domain included in  $\mathbb{R}$  and the codomain included in  $\mathbb{R}$ , you can represent those points in a graph and visualize the properties of your function graphically. This might help you to make up your mind about a question.

**Here sample questions I could ask you about graph :** Suppose that the graph below is the graph of a function  $f$



1. Why is this really the graph of a function ?

**Solution :** Since for each abscissa there is a unique point with this abscissa belonging to the graph.

2. Graphically, determine the image of 0 and 2 by  $f$ .

**Solution :** The graph of the function pass through  $(0, 5)$ , thus the image of 0 is 5 and the graph pass through  $(2, 5)$  thus the image of 2 is also 5.

3. Graphically, determine all the preimages of 1, 3, 5 and 6.

**Solution :** The preimages of 1 are the abscissas of the points of intersection of the horizontal lines passing through the point  $(0, 1)$  with the graph of the function  $f$ , thus the preimages of 1 constitute the interval  $[0, 2]$ . The preimages of 3 are the abscissas of the points of intersection of the horizontal lines passing through the point  $(0, 3)$  with the graph of the function  $f$ , thus the preimages of 3 are  $-2$  and  $4$ . The preimages of 5 are the abscissas of the points of intersection of the horizontal lines passing through the point  $(0, 5)$  with the graph of the function  $f$ , thus the preimages of 5 is  $-1$ . The preimages of 6 are the abscissas of the points of intersection of the horizontal lines passing through the point  $(0, 6)$  with the graph of the function  $f$ , thus there is no preimage for 6.

In conclusion, graphically the image of a point  $x$  in the domain is the ordinate of the point belonging to a graph whose abscissa is  $x$  and the preimages of a  $y$  in the codomain are the abscissas of the points of intersection of the horizontal lines passing through the point  $(0, y)$  with the graph of the function  $f$ .

## Range, image and preimage of a set

If you understood, some of the values of the codomain might not have a preimage. If you want to find out precisely what are the elements that have a preimage you can compute what we call the range.

The **range of a function**  $f : A \rightarrow B$ , denoted by  $\text{Range}(f)$ , is the set of all the possible image in the codomain for values  $x$  varying through all the domain. In mathematical words,

$$\text{Range}(f) = \{f(x) : x \in A\}$$

**Here sample questions I could ask you about range :**

*Let*

$$\begin{array}{ccc} f : & \mathbb{R} & \rightarrow \mathbb{R} \\ & x & \mapsto x^2 \end{array}$$

*Compute the range of this function.*

**Solution :** *The range of this function is*

$$\text{Range}(f) = \{f(x) : x \in \mathbb{R}\} = \{x^2 : x \in \mathbb{R}\} = \mathbb{R}^+$$

*In other words, only positive numbers can have preimage via the function  $f$ .*

If you want to know not all the images for the all domain but only for certain values of your domain. You will compute what we call an image set.

Let  $S$  be a subset of  $A$  the domain of  $f$  (we write  $S \subset A$ ). The **image of the set**  $S$  denoted by  $f(S)$  is the set of all the possible images of elements  $x$  varying through all the the set  $S$ . In mathematical words, we write

$$f(S) = \{f(x) \in B : x \in S\}$$

Note that  $f(S)$  is a subset of the codomain.

**Here sample questions I could ask you about images of a set :**

*Let*

$$\begin{array}{ccc} f : & \mathbb{R} & \rightarrow \mathbb{R} \\ & x & \mapsto x^2 \end{array}$$

*Compute  $f([0,1])$  and  $f(\{1,2\})$ .*

**Solution :** *By definition,*

$$f([0,1]) = \{f(x) \in \mathbb{R} : x \in [0,1]\} = \{x^2 : x \in [0,1]\} = [0,1]$$

*Indeed,  $0 \leq x \leq 1$  if and only if  $0 \leq x^2 = x \times x \leq 1$ .*

*In other words, when  $x$  varies through  $[0,1]$ ,  $f(x)$  also varies through  $[0,1]$  and reaches each of those value.*

$$f(\{1,2\}) = \{f(x) \in \mathbb{R} : x \in \{1,2\}\} = \{x^2 : x \in \{1,2\}\} = \{1^2, 2^2\} = \{1,4\}$$

*In other words, when  $x$  varies through  $\{1,2\}$ , that is  $x$  takes the value either 1 or 2,  $f(x)$  takes the value 1 and 4.*

Let  $T$  be a subset of  $B$  the codomain of  $f$  (we write  $T \subset B$ ). The **preimage of the set**  $S$  denoted by  $f^{-1}(T)$  is the set of all the possible preimages of elements  $x$  varying through all the the set  $S$ . In mathematical words, we write

$$f^{-1}(T) = \{x \in A : f(x) \in T\}$$

Note that  $f^{-1}(T)$  is a subset of the domain.

**Here sample questions I could ask you about preimages of a set :**

Let

$$\begin{array}{rcl} f: \mathbb{R} & \rightarrow & \mathbb{R} \\ x & \mapsto & x^2 \end{array}$$

Compute  $f^{-1}([0,1])$  and  $f^{-1}(\{1,2\})$ .

**Solution :** By definition,

$$f^{-1}([0,1]) = \{x \in \mathbb{R} : f(x) \in [0,1]\} = \{x \in \mathbb{R} : x^2 \in [0,1]\} = [-1,1]$$

Indeed,  $0 \leq x^2 \leq 1$  if and only if  $x^2 \geq 0$  and  $x^2 \leq 1$ , the first one is always true since a square is always positive, and the second is equivalent to  $x^2 - 1 = (x-1)(x+1) \leq 0$  doing a sign table you find that this is equivalent to  $x \in [-1,1]$ , which justify the line above.

In other words, if I take any elements in  $[0,1]$  and what to find all the preimages for all those elements I get that all the elements  $[-1,1]$  are preimages for elements within the interval  $[0,1]$ .

$$f^{-1}(\{1,2\}) = \{x \in \mathbb{R} : x^2 \in \{1,2\}\} = \{-\sqrt{2}, -1, 1, \sqrt{2}\}$$

In other words, you are trying to find the set of all preimages for elements in  $\{1,2\}$ , so you only need to find all the preimages for 1 and 2, that is the real numbers  $x$  such that  $x^2 = 1$  or  $x^2 = 2$  and you find out that those are precisely  $\{-\sqrt{2}, -1, 1, \sqrt{2}\}$ .

## Onto function

Let  $f : A \rightarrow B$ .

When the range is precisely equal to the codomain we say that the function is **onto**. In other words, a function is onto if any element of the codomain has at least a preimage. Mathematically we write

$$\forall y \in B, \exists x \in A : f(x) = y$$

A function is **not onto** as soon as you can find ONE element in the codomain that has no preimage. Mathematically we write

$$\exists y \in B, \forall x \in A : f(x) \neq y$$

**Here sample questions I could ask you about onto :**

1. Let

$$\begin{array}{ccc} f : \mathbb{R} & \rightarrow & \mathbb{R} \\ x & \mapsto & x^2 \end{array}$$

Prove that  $f$  is not onto  $\mathbb{R}$ .

**Solution :** Let  $y = -1 \in \mathbb{R}$ , if  $f$  was onto we will have a  $x \in \mathbb{R}$  such that  $f(x) = x^2 = -1$ . But we all know that a square is always positive. Thus that is not possible such a  $x$  will never exist proving that  $f$  is not onto.

2. Let

$$\begin{array}{ccc} g : \mathbb{R} & \rightarrow & \mathbb{R} \\ x & \mapsto & 2x + 3 \end{array}$$

Prove that  $g$  is onto  $\mathbb{R}$ .

**Solution :** Let  $y \in \mathbb{R}$ . We are looking for a  $x \in \mathbb{R}$  such that  $g(x) = y$ . (Note : If  $y$  is taken before we looking for  $x$  thus  $x$  can depend on  $y$ . The existence of this  $x$  possibly in term of  $y$  for any  $y$  will prove that the function is onto.) Now, if we had such a  $x$  existing we should have  $2x + 3 = y$  meaning that  $x$  is given by  $x = \frac{y-3}{2}$ . Now let verify that all is fine. We pick an arbitrary  $y$  in the codomain  $\mathbb{R}$ , and choose  $x = \frac{y-3}{2}$  this belong to the domain  $\mathbb{R}$ , moreover

$$g(x) = g\left(\frac{y-3}{2}\right) = 2\frac{y-3}{2} + 3 = y.$$

Thus  $x$  is a preimage of  $y$  and we have proven that  $g$  is onto.

## One to one function

Let  $f : A \rightarrow B$ .

When every element of the codomain has at most one preimage we say that the function is one-to-one. Mathematically we write

$$\forall x_1, x_2 \in A, (f(x_1) = f(x_2) \Rightarrow x_1 = x_2)$$

(In english, this just means that if you take two elements in your domain that have the same image your function can be one to one if and only if those two elements are the same. Since if they were not the same you would have two different preimages for the same value in the codomain which is not allowed by the definition of a one-to-one function)

A function is **not one-to-one** as soon as you can find two distinct elements in the domain with the same image. Mathematically we write

$$\exists x_1, x_2 \in A : x_1 \neq x_2 \text{ and } f(x_1) = f(x_2)$$

**Here sample questions I could ask you about one to one :**

1. Let

$$\begin{array}{ccc} f : & \mathbb{R} & \rightarrow \mathbb{R} \\ & x & \mapsto x^2 \end{array}$$

*Prove that  $f$  is not one to one.*

**Solution :** Let  $x_1 = -1$  and  $x_2 = 1$  elements of the domain  $\mathbb{R}$ . Clearly,  $x_1 \neq x_2$  but  $f(x_1) = (-1)^2 = 1$  and  $f(x_2) = 1^2 = 1$ . Thus  $f(x_1) = f(x_2)$ . We have found two distinct points in the domain that have the same image, thus  $f$  is not one-to-one.

2. Let

$$\begin{array}{ccc} g : & \mathbb{R} & \rightarrow \mathbb{R} \\ & x & \mapsto 2x + 3 \end{array}$$

*Prove that  $f$  is onto  $\mathbb{R}$ .*

**Solution :** Let  $x_1, x_2 \in \mathbb{R}$ , suppose  $g(x_1) = g(x_2)$ . You will need to prove that  $x_1 = x_2$ .

Since  $f(x_1) = f(x_2)$  by assumption we have  $2x_1 + 3 = 2x_2 + 3$  thus  $2x_1 = 2x_2$  if we subtract both side of the previous equality 3 and finally  $x_1 = x_2$  dividing the previous equality by 2. Thus we have proven that for any  $x_1, x_2 \in \mathbb{R}$ , if  $g(x_1) = g(x_2)$  then  $x_1 = x_2$ . Thus  $g$  is one-to-one.

## Bijection and inverse

Let  $f : A \rightarrow B$ .

We say the  $f$  is a bijection if it is both one-to-one and onto. Equivalently, that means that for each element in the codomain there exists exactly one unique preimage in the domain. Mathematically, we write

$$\forall y \in B, \exists! x \in A : f(x) = y$$

A function is **not bijective** if you can prove that it is either not onto or one-to-one. (You need only one of the two to fail to be able to conclude that your function is not bijective.)

When a function is bijective, it admits an inverse function that is a function whose domain is the codomain of  $f$  and codomain is the domain of  $f$  and the rule send an  $y$  to its unique preimage. (DO NOT SPEAK ABOUT INVERSES IF YOU DO NOT HAVE PROVEN THAT YOUR FUNCTION IS A BIJECTION) Mathematically, we denote the inverse of  $f$ ,  $f^{-1}$  and is defined as

$$\begin{array}{lll} f : & B & \rightarrow A \\ y & \mapsto & x \text{ with } y = f(x) \end{array}$$

The inverse function is also characterized as the function  $f^{-1} : B \rightarrow A$  such that

$$f \circ f^{-1} = Id_B \text{ and } f^{-1} \circ f = Id_A$$

where

$$\begin{array}{lll} Id_B : & B & \rightarrow B \\ x & \mapsto & x \end{array}$$

and

$$\begin{array}{lll} Id_A : & A & \rightarrow A \\ x & \mapsto & x \end{array}$$

**Here sample questions I could ask you about bijections :**

1. Let

$$\begin{array}{lll} f : & \mathbb{R} & \rightarrow \mathbb{R} \\ x & \mapsto & x^2 \end{array}$$

*Prove that  $f$  is bijective.*

**Solution :**  $f$  is not bijective sin.

2. Let

$$\begin{array}{lll} g : & \mathbb{R} & \rightarrow \mathbb{R} \\ x & \mapsto & 2x + 3 \end{array}$$

*Prove that  $f$  is bijective  $\mathbb{R}$  and describe the inverse.*

**Solution :** We have prove before that this function is both one-to-one and onto. Also, we have seen when we have proven it was onto that for each  $y \in \mathbb{R}$ ,  $x = \frac{y-3}{2}$  was a preimage. Since now we know it is bijective, we know also that this has to be the only preimage of  $y$  since by definition bijective means that for any elements in the codomain there is a unique preimage in the domain. In conclusion, the inverse function of  $g$  is given by

$$\begin{array}{lll} g^{-1} : & \mathbb{R} & \rightarrow \mathbb{R} \\ x & \mapsto & \frac{x-3}{2} \end{array}$$

*Since we can double check that and make sure our answer is correct. Let's do it :*

$$g \circ g^{-1}(x) = g(g^{-1}(x)) = f\left(\frac{x-3}{2}\right) = 2\frac{x-3}{2} + 3 = x$$

thus

$$g \circ g^{-1} = Id_{\mathbb{R}}$$

and

$$g^{-1} \circ g(x) = g^{-1}(g(x)) = g^{-1}(2x+3) = \frac{2x+3-3}{2} = x$$

thus

$$g^{-1} \circ g = Id_{\mathbb{R}}$$

We can sleep tight, it seems that our answer is correct.

3. Are

$$\begin{array}{ccc} g: & \mathbb{R} & \rightarrow & \mathbb{R} \\ & x & \mapsto & 3x+3 \end{array}$$

and

$$\begin{array}{ccc} h: & \mathbb{R} & \rightarrow & \mathbb{R} \\ & x & \mapsto & x-3 \end{array}$$

inverses of each other?

**Solution :** Well, let see if they were we would have  $g \circ h = Id_{\mathbb{R}}$  and  $h \circ g = Id_{\mathbb{R}}$ . That is easy to check. Look, let  $x = 1$

$$g \circ h(1) = g(h(1)) = g(1-3) = g(-2) = 3(-2) + 3 = -3 \neq 1$$

thus  $g \circ h \neq Id_{\mathbb{R}}$  and this one  $x = 1$  is enough to conclude that those functions are not inverses of each other.

## Even or Odd

A function  $f : A \rightarrow B$  is **even** if for every  $x \in A$ ,  $f(-x) = f(x)$ .

A function  $f : A \rightarrow B$  is **not even** if there is  $x \in A$ ,  $f(-x) \neq f(x)$ .

A function  $f : A \rightarrow B$  is **odd** if for every  $x \in A$ ,  $f(-x) = -f(x)$ .

A function  $f : A \rightarrow B$  is **not odd** if for every  $x \in A$ ,  $f(-x) \neq -f(x)$ .

Here sample questions I could ask you about even and odd :

1. Let

$$\begin{array}{ccc} f : \mathbb{R} & \rightarrow & \mathbb{R} \\ x & \mapsto & x^2 \end{array}$$

Prove that  $f$  is even  $\mathbb{R}$ .

**Solution :** Let  $x \in \mathbb{R}$ ,  $f(-x) = (-x)^2 = x^2 = f(x)$ . (**Advise :** When proving an equality either you go from a one side to the other or you compute both side separatly until you get the same answer somehow. Never prove an equality starting with the equality it is non sense since it is what you want to prove. Another thing is that if you choose the first option you can start the left hand side but also with the right hand side, think of which one will be the easiest for your computations.)

2. Let

$$\begin{array}{ccc} g : \mathbb{R} & \rightarrow & \mathbb{R} \\ x & \mapsto & 2x + 3 \end{array}$$

Prove that  $g$  is neither odd or even  $\mathbb{R}$ .

**Solution :** Let  $x = 1$ .  $g(-1) = 2(-1) + 3 = 1$  and  $g(1) = 2 \times 1 + 3 = 5$ . Thus  $g(-1) \neq g(1)$  and  $g$  is not even but also  $g(-1) \neq -f(1)$  proving that  $g$  is not odd.

3. Let

$$\begin{array}{ccc} k : \mathbb{R} & \rightarrow & \mathbb{R} \\ x & \mapsto & x^3 \end{array}$$

Prove that  $k$  is odd  $\mathbb{R}$ .

**Solution :** Let  $x \in \mathbb{R}$ ,  $k(-x) = (-x)^3 = -x^3 = -f(x)$ . Thus  $f$  is odd.

## Increasing and decreasing function

Let  $I \subset A$ .

A function  $f : A \rightarrow B$  is **increasing over  $I$**  if for every  $x_1, x_2 \in I$ , if  $x_1 \leq x_2$  then  $f(x_1) \leq f(x_2)$ .

A function  $f : A \rightarrow B$  is **strictly increasing over  $I$**  if for every  $x_1, x_2 \in I$ , if  $x_1 < x_2$  then  $f(x_1) < f(x_2)$ .

A function  $f : A \rightarrow B$  is **not increasing over  $I$**  if you can find two  $x_1, x_2 \in I$ , with  $x_1 \leq x_2$  but  $f(x_1) > f(x_2)$ .

A function  $f : A \rightarrow B$  is **strictly increasing over  $I$**  if for every  $x_1, x_2 \in I$ , if  $x_1 < x_2$  then  $f(x_1) \geq f(x_2)$ .

A function  $f : A \rightarrow B$  is **decreasing over  $I$**  if for every  $x_1, x_2 \in I$ , if  $x_1 \leq x_2$  then  $f(x_1) \geq f(x_2)$ .

A function  $f : A \rightarrow B$  is **strictly decreasing over  $I$**  if for every  $x_1, x_2 \in I$ , if  $x_1 < x_2$  then  $f(x_1) < f(x_2)$ .

A function  $f : A \rightarrow B$  is **not decreasing over  $I$**  if you can find two  $x_1, x_2 \in I$ , with  $x_1 \leq x_2$  but  $f(x_1) < f(x_2)$ .

A function  $f : A \rightarrow B$  is **not decreasing over  $I$**  if for every  $x_1, x_2 \in I$ , if  $x_1 < x_2$  then  $f(x_1) \leq f(x_2)$ .

**Here sample questions I could ask you about increasing and decreasing :**

1. Let

$$\begin{array}{ccc} f : & \mathbb{R} & \rightarrow \mathbb{R} \\ & x & \mapsto x^2 \end{array}$$

Prove that  $f$  is neither increasing nor decreasing over  $\mathbb{R}$ .

**Solution :** Let  $x_1 = 1$  and  $x_2 = 2$  real numbers, we have  $x_1 \leq x_2$ , but  $f(x_1) = 1^2 = 1$  and  $f(x_2) = 2^2 = 4$ , so that  $f(x_1) < f(x_2)$ . Thus  $f$  is not decreasing. Let  $x_1 = -2$  and  $x_2 = -1$  real numbers, we have  $x_1 \leq x_2$ , but  $f(x_1) = (-2)^2 = 4$  and  $f(x_2) = (-1)^2 = 1$ , so that  $f(x_1) > f(x_2)$ . Thus  $f$  is not increasing. In conclusion,  $f$  is neither decreasing nor increasing.

2. Let

$$\begin{array}{ccc} g : & \mathbb{R} & \rightarrow \mathbb{R} \\ & x & \mapsto -2x + 3 \end{array}$$

Prove that  $g$  is decreasing  $\mathbb{R}$ .

**Solution :** Let  $x_1, x_2 \in \mathbb{R}$ , such that  $x_1 \leq x_2$ , then  $-2x_1 \geq -2x_2$ . Finally,  $-2x_1 + 3 \leq -2x_2 + 3$ , proving that  $f(x_1) \geq f(x_2)$ . We have proven this for arbitrary  $x_1, x_2$  real numbers thus,  $g$  is decreasing

# Polynomials

A **real polynomial function** is a function  $p : I \rightarrow J$  with  $I, J \subset \mathbb{R}$  such that

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where  $a_i \in \mathbb{R}$  and  $n \in \mathbb{N}$ . The  $a_i$ 's are called coefficient,  $a_0$  is the constant coefficient and  $n$  is the degree of  $p(x)$ . By convention, we say that  $p(x) = 0$  has no degree or degree  $-\infty$ .

A **real rational function** is a function  $f : I \rightarrow J$  with  $I, J \subset \mathbb{R}$  such that

$$f(x) = \frac{p(x)}{q(x)}$$

and  $q(x) \neq 0$  for any  $x \in I$ .

**Here sample questions I could ask you about polynomials :**

1. Is  $p : \mathbb{R} \rightarrow \mathbb{R}$  defined by the rule  $p(x) = \sqrt{x^2}$  a polynomial ?

**Solution :** No it is not a polynomial  $p(x) = |x|$  and  $|x|$  is not a power function.

2. Is  $p : \mathbb{R}^+ \rightarrow \mathbb{R}$  defined by the rule  $p(x) = \sqrt{x^2}$  a polynomial ?

**Solution :** Yes, since  $p(x) = x$ .

3. Is  $p : \mathbb{R} \rightarrow \mathbb{R}$  defined by the rule  $p(x) = 3x^3 + x^2 + 3$  a polynomial ? If so, what is its degree ? constant coefficient ?

**Solution :**  $p$  is a polynomial by definition. Its degree is 3 and the constant coefficient is also 3.